Computing Equilibria in Static Games of Incomplete Information Using the All-Solution Homotopy*

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Abstract

In this paper we analyze the application of numerical continuation methods to compute equilibria in a class of static games of incomplete information. Computation of equilibria in such games is usually a challenging task because the sets of equilibria can have complex configurations. The all-solution homotopy approach allows one to use a set of existing path-tracking tools to compute asymptotic approximations to the sets of equilibria at a relatively low computational cost. We provide some results that these tools can provide asymptotically good approximations to sets of equilibria in games. We illustrate our findings using a simple example of an entry game with linear payoffs. We also discuss a possible approach to estimating games with multiple equilibria using a pre-computed set of equilibria.

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1 Introduction

During the past decade, one of the most active areas of research in industrial organization has been the intersection of econometrics and game theory. Starting with Bjorn and Vuong (1984), researchers have developed empirical models of discrete games. In these models, the dependent variable has been a discrete choice (e.g. enter or don't enter). The payoffs are determined as in a random utility model such as a multinomial logit. They depend on exogenous variables, parameters and random preference shocks. However, unlike a logit, the utilities also depend on the actions of "other" agents. This has developed into a large literature with contributions by Bresnahan and Reiss (1991a), Berry (1994), Tamer (2003a), Seim (2006), Aguirregabiria and Mira (2002) and many others.

There are two modeling approaches in this literature. In the first approach, the error terms in the discrete choice model are common knowledge and in the second they are private information. In the

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former case, the models can generate a large number of possible equilibrium. Many researchers, such as Tamer (2003a), Pakes, Porter, Ho, and Ishii (2007), Shaikh and Romano (2005) and many others have advocated bounds estimators. This allows the researchers to work with necessary conditions from the models and not worry about equilibrium selection. In the later case, when error terms are private information, estimation is frequently done in two stages. For example, in static models, Bajari, Hong, Krainer, and Nekipelov (2010) show that the estimators involve simple extensions of static logit models.

In this paper, we propose new numerical algorithms to compute the entire set of equilibrium for private information games. These algorithms involve the application of the all solutions homotopy. A homotopy is a numerical method that is commonly used to solve a nonlinear system of equations. The idea behind a homotopy is quite simple and intuitive. The researcher starts with a "simple" problem that is easy to solve. The researcher continuously deforms this simple problem into the problem of interest by taking a weighted average of the two problems. Under suitable conditions, the homotopy method is able to trace a path that leads a solution of the initial system to a solution of final system of interest.

The applicability of homotopy methods to equilibrium computations did not remain unnoticed in the literature. For instance, Borkovsky, Doraszelski, and Kryukov (2008) provide a user guide for equilibrium computation in games using homotopy techniques. And Besanko et al. (2007) and Schmedders and Judd (2005) use homotopy techniques to compute equilibria in a dynamic game. The contribution of our paper is that we propose to use a complex-valued all-solution homotopy that is guaranteed to provide asymptotic coverage for the equilibrium sets in games of incomplete information. To our knowledge, such a possibility has not been considered before. For example, Schmedders and Judd (2005) focuses on equilibria in polynomial systems. Note that Watson (1990) also considers homotopy methods in the complex plane.

Zangwill and Garcia (1980) describe the all solutions homotopy. This method is commonly used to find all the solutions to a polynomial system of degree n. In our paper, we adapt the all solutions homotopy to find as many roots as possible to private information games. As we show, an equilibrium to a private information game is a fixed point that can be represented in closed form. The system of equations characterizes the choice probabilities generated by a logit model. In section 4.2 we show that theoretically, if the degree of the homotopy increases to infinity without bound, all the solutions of private information games can be found under suitable regularity conditions. However, the validity of finding all solutions by continuously increasing the order of the homopoty is mostly a theoretical results, since we are not able to provide an algorithmic stop rule to automate the determination of the order of the homotopy polynomial. In practice, a course of suggested action is to increase the degree of initial polynomial system until the method can not find any more equilibria. One computational advantage of this approach is that it is easy to parallelize on modern computers.

We report the results from a computational exercise in which we engage in computing the equilibrium set for a simulated set of incomplete information games. We have several surprising findings. First, the number of equilibria in incomplete information games appears to decrease as the complexity of the game (e.g., the number of players and actions) increases. This is in contrast to complete information games, in which the number of equilibrium typically increases at an exponential rate. See for example McKelvey, McLennan, and Turocy (2006), McKelvy and McLennan (1996), Datta (2010) and Kalaba and Tesfatsion (1991), who use homotopy to solve finite games whose equilibria can be represented through a system of polynomial equations. This finding is interesting for a couple of reasons. First, estimation approaches in incomplete information games have been criticized for assuming that the econometrician can consistently estimate agent's beliefs. Researchers have argued that this may not be possible if agents flip back and forth between different equilibria in the data. Our research shows that the vast majority of our games have only one equilibrium. This suggests that this assumption may not be a bad empirical approximation. We do caution that this is merely an example from a numerical exercise. Since Harsanyi has shown that mixed strategy equilibria of a complete information games are limits of sequences of incomplete information games, our finding might not generalize beyond the specific examples that we consider. Second, our research shows that the equilibrium set of outcomes strongly depends on the incomplete information assumption. The choice between an incomplete information and a complete information model can make very different predictions about behavior since they have very different equilibrium sets. In applied work, there is no rigorous criteria for choosing between these methods. Our work shows that developing such a criteria may be very important for ensuring the credibility of estimates and counterfactual simulations based on these models.

The point of our paper is to show that the assumption of incomplete information generates a profound implication in terms of the properties of the equilibrium set, such as the number of equilibrium and the predictions of the model. Which model is right is ultimately an empirical question. The literature needs to develop hypothesis tests which are capable of reliably distinguishing between different models and testing each specification. This is beyond the scope of this paper. The point here is that our computational tools show us that the incomplete information assumption can matter a great deal in empirical research.

2 The model

In the model, there are a finite number of players, $i = 1, ..., n$ and each player simultaneously chooses an action $a_i \in \{0, 1, \dots, K\}$ out of a finite set. We restrict players to have the same set of actions for notational simplicity. However, all of our results will generalize to the case where all players have different finite sets of actions. Let $A = \{0, 1, \dots, K\}^n$ denote the vector of possible actions for all players and let $a = (a_1, ..., a_n)$ denote a generic element of A. As is common in the literature, we let $a_{-i} = (a_1, ...a_{i-1}, a_{i+1}, ..., a_n)$ denote a vector of strategies for all players, excluding player i. We will abstract from mixed strategies since in our model, with probability one each player will have a unique best response.

Let $s_i \in S_i$ denote the state variable for player i. Let $S = \Pi_i S_i$ and let $s = (s_1, ..., s_n) \in S$ denote a vector of state variables for all n players. We will assume that s is common knowledge to all players in the game and in our econometric analysis, we will assume that s is observable to the econometrician. The state variable is assumed to be a real valued vector, but S_i is not required to be a finite set. Much of the previous literature assumes that the state variables in a discrete games lie in a discrete set. While this assumption simplifies the econometric analysis of the estimator and identification, it is a strong assumption that may not be satisfied in many applications.

For each agent, there are also $K+1$ preference shocks which we label as $\epsilon_i(a_i)$ which are private information to each agent. These preference shocks are distributed i.i.d. across agents and actions. Let ϵ_i denote the $1 \times (K+1)$ vector of the individual $\epsilon_i(a_i)$. The density of $\epsilon_i(a_i)$ will be denoted as $f(\epsilon_i(a_i))$. However, we shall sometimes simplify the notation and denote the density for $\epsilon_i = (\epsilon_i(0), ..., \epsilon_i(K))$ as $f(\epsilon_i)$.

The period utility function for player i is:

$$u_i(a, s, \epsilon_i) = \pi_i(a_i, a_{-i}, s) + \epsilon_i(a_i). \tag{2.1}$$

The utility function in our model is similar to a standard random utility model such as a multinomial logit. Each player i receives a stochastic preference shock, $\epsilon_i(a_i)$, for each possible action a_i. In many applications, this will be drawn from an extreme value distribution as in the logit model. In the literature, the preference shock is alternatively interpreted as an unobserved state variable (see Rust (1994)). Utility also depends on the vector of state variables s and actions a through $\Pi_i(a_i, a_{-i}, s; \theta)$. For example, in the literature, this part of utility is frequently parameterized as a simple linear function of actions and states. Unlike a standard discrete choice model, however, note that the actions a_{-i} of other players in the game enter into i's utility. A standard discrete choice model typically assumes that agents i act in isolation in the sense that a_{-i} is omitted from the utility function. In many applications, this is an implausible assumption.

In this model, player i's decision rule is a function $a_i = \Delta_i(s, \epsilon_i)$. Note that i's decision does not depend on the ϵ_{-i} since these shocks are private information to the other $-i$ players in the game and, hence, are unobservable to i. Define $\sigma_i(a_i|s)$ as:

$$\sigma_i(a_i = k|s) = \int 1\left\{\Delta_i(s, \epsilon_i) = k\right\} f(\epsilon_i) d\epsilon_i.$$

In the above expression, $1\left\{\Delta_i(s, \epsilon_i) = k\right\}$ is the indicator function that player $\acute{\imath}$'s action is k given the vector of state variables (s, ϵ_i). Therefore, $\sigma_i(a_i = k|s)$ is the probability that i chooses action k conditional on the state variables s that are public information. We will define the distribution of a given s as $\sigma(a|s) = \Pi_{i=1}^{n} \sigma(a_i|s)$.

Next, define $U_i(a_i, s, \epsilon_i; \theta)$ as:

$$\begin{aligned} U_i(a_i, s, \epsilon_i) =& \sum_{a_{-i}} \pi_i(a_i, a_{-i}, s) \sigma_{-i}(a_{-i}|s) + \epsilon_i(a_i) \\ \text{where } \sigma_{-i}(a_{-i}|s) =& \Pi_{j \neq i} \sigma_j(a_j|s). \end{aligned} \tag{2.2}$$

In (2.2), $U_i(a_i, s, \epsilon_i; \theta)$ is player i's expected utility from choosing a_i when the vector of parameters is θ. Since i does not know the private information shocks, ϵ_j for the other players, i's beliefs about their actions are given by $\sigma_{-i}(a_{-i}|s)$. The term $\sum_{a_{-i}} \pi_i(a_i, a_{-i}, s, \theta)\sigma_{-i}(a_{-i}|s)$ is the expected value of $\pi_i(a_i, a_{-i}, s; \theta)$, marginalizing out the strategies of the other players using $\sigma_{-i}(a_{-i}|s)$. The structure of payoffs in (2.2) is quite similar to standard random utility models, except that the probability distribution over other agents' actions enter into the formula for agent i's utility. Note that if the error term has an atomless distribution, then player i's optimal action is unique with probability one. This is an extremely convenient property and eliminates the need to consider mixed strategies as in a standard normal form game.

We also define the deterministic part of the expected payoff as

$$\pi_i\left(a_i, s\right) = \sum_{a_{-i}} \pi_i(a_i, a_{-i}, s)\sigma_{-i}(a_{-i}|s). \tag{2.3}$$

It follows immediately then that the optimal action for player i satisfies:

$$\sigma_i(a_i|s) = Prob\left\{\epsilon_i | \pi_i(a_i, s) + \epsilon_i(a_i) > \pi_i(a_j, s) + \epsilon_i(a_j) \text{ for } j \neq i.\right\} \tag{2.4}$$

A common convenient assumption regarding the distribution of $\epsilon_i(\cdot)$ is that they are i.i.d. extreme value distributed across actions. Our computational methodology will be tailored to this case. However, our results generalize to the case of arbitrary shock distribution with continuous density and unbounded support via a transformation of the cumulative distribution function.

3 The Game of Entry

For expositional clarity, consider a simple example of a discrete game. Perhaps the most commonly studied example of a discrete game in the literature is a static entry game (see Bresnahan and Reiss (1991a), Bresnahan and Reiss (1991b), Berry (1992), Tamer (2003b), Ciliberto and Tamer (2009), Manuszak and Cohen (2004)). In the empirical analysis of entry games, the economist typically has data on a cross section of markets and observes whether a particular firm i chooses to enter a particular market. In Berry (1992) and Ciliberto and Tamer (2009), for example, the firms are major U.S. airlines such as American, United and Northwest and the markets are large, metropolitan airports. The state variables, s_i, might include the population in the metropolitan area surrounding the airport and measures of an airline's operating costs. Let $a_i = 1$ denote the decision to enter a particular market and $a_i = 0$ denote the decision not to enter the market. In many applications, $\pi_i(a_i, a_{-i}, s) = \pi_i(a_i, a_{-i}, s; \theta)$ is assumed to be a linear function of a vector of parameters, e.g.:

$$\pi_i\left(a_i, a_{-i}, s; \theta\right) = \begin{cases} s' \cdot \beta + \delta \sum_{j \neq i} 1\left\{a_j = 1\right\} & \text{if} \quad a_i = 1, \\ 0 \quad \text{if} \quad a_i = 0 \end{cases} \tag{3.5}$$

In equation (3.5), $\theta = (\beta, \delta)$, and the mean utility from not entering is set equal to zero.[2] The term δ measures the influence of j's choice on i's entry decision. If profits decrease from having another firm enter the market then $\delta < 0$. The parameters β measure the impact of the state variables on $\pi_i(a_i, a_{-i}, s; \theta)$. The other functions of the model should also be considered as functions of the relevant parameters.

The random error terms $\epsilon_i(a_i)$ are thought to capture shocks to the profitability of entry that are private information to firm i. Suppose that the error terms are distributed extreme value. Then, utility maximization by firm i implies that for $i = 1, ..., n$

$$\sigma_i(a_i = 1|s) = \frac{\exp\left(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s)\right)}{1 + \exp\left(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s)\right)} \tag{3.6}$$

In the system of equations above, applying the formula in equation (2.3) implies that $\pi_i\left(a_i, s; \theta\right) = s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s)$. Since the error terms are distributed extreme value, equation (2.4) implies that the choice probabilities, $\sigma_i(a_i = 1|s)$ take a form similar to a single agent multinomial logit model. We note in passing that it can easily be shown using Brouwer's fixed point theorem that an equilibrium to this model exists for any finite s (see McKelvey and Palfrey (1995))).

The convenient representation of equilibrium in equation (3.6) can be exploited in econometric analysis. Suppose that the econometrician observes $t = 1, ..., T$ repetitions of the game and it is assumed that in each repetition the players get new draws of private shocks. Let $a_{i,t}$ denote the entry decision of firm i in repetition t and let the value of the state variables be equal to s_t. By observing entry behavior in a large number of markets, the econometrician could form a consistent estimate $\widehat{\sigma}_i(a_i = 1|s)$ of $\sigma_i(a_i = 1|s)$ for $i = 1, ..., n$. In an application, this simply boils down to flexibly estimating the probability that a binary response, a_i, is equal to one, conditional on a given set of covariates. This could be done using any one of a number of standard techniques. Note that this approach is only feasible observationally equivalent markets always play the same equilibrium.

Given first stage estimates of $\widehat{\sigma}_i(a_i = 1|s)$, we could then estimate the structural parameters of the payoff, β and δ, by maximizing a pseudo-likelihood function using (3.6). There are two attractive features of this strategy. The first is that it is not demanding computationally. First stage estimates of choice probabilities could be done using a strategy as simple as a linear probability model. The computational burden of the second stage is also light since we only need to estimate a logit model. A second attractive feature is that it allows us to view a game as a generalization of a standard discrete choice model. Thus, techniques from the voluminous econometric literature on discrete choice models can be imported into the study of strategic interaction. While the example considered

[2] Possible normalizations are formally discussed in Bajari, Hong, Krainer, and Nekipelov (2010).

above is simple, it nonetheless illustrates many of the key ideas that will be essential in what follows.

We can also see a key problem with identification in the simple example above. Both the first stage estimates $\widehat{\sigma}_i(a_i = 1|s)$ and the term $s' \cdot \beta$ depend on the vector of state variables s. This suggests that we will suffer from a collinearity problem when trying to separately identify the effects of β and δ on the observed choices. The standard solution to this type of problem in many settings is to impose an exclusion restriction. Suppose, for instance, a firm specific productivity shock is included in s. In most oligopoly models, absent technology spillovers, the productivity shocks of firms $-i$ would not directly enter into firm i's profits. These shocks only enter indirectly through the endogenously determined actions of firms $-i$, e.g. price, quantity or entry decisions. Therefore, if we exclude the productivity shocks of other firms from the term $s' \cdot \beta$, we would no longer suffer from a collinearity problem.

4 Computing Models with Multiple Equilibria.

In the previous sections, we have either assumed that the model has a unique equilibrium (which can be the case, for example, for a linear probability interaction model), or that only a single equilibrium outcome out of several possible multiple equilibria is being observed in the data set. However, in many static game models, multiple equilibria are possible. The importance of multiple equilibria in empirical research is emphasized by many authors, including Brock and Durlauf (2001) and Sweeting (2008). In the rest of this manuscript we present a method for estimating parametric models of interactions in the presence of possible multiple equilibria.

In the previous sections we have considered a model with known distribution $F(\epsilon_i)$ of the error terms and a parametric model for the mean utility functions $\pi_i\left(a_i, a_{-i}, \theta\right)$. At every possible parameter value θ, given the known distribution $F\left(\epsilon_i\right)$, equations (2.3) and (2.4) defined a fixed point mapping in the conditional choice probabilities:

$$\sigma_i(a_i|s) = \Gamma_i\left(\sum_{-i} \sigma_{a_{-i}}\left(a_{-i}|s\right)\left[\pi_i\left(a_i, a_{-i}, s; \theta\right) - \pi_i\left(k, a_{-i}, s; \theta\right)\right], k = 1, \ldots, K, k \neq a_i\right). \tag{4.7}$$

For example, under the linear mean utility specification (3.5), this system of fixed point mappings in the choice probabilities takes the form of

$$\sigma_i(a_i = 1|s) = \Gamma_i\left(s'\theta + \delta \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s)\right),\ i = 1, \ldots, n. \tag{4.8}$$

In previous section, we have assumed that either there is a unique solution to this system of fixed mapping with $K \times n$ equations and $K \times n$ unknown variables

$$\sigma_i\left(a_i|s\right), \forall a_i = 1, \ldots, K, i = 1, \ldots, n,$$

or that only one particular fixed point of this system gets realized in the observed data. However, this system of fixed point mapping can potentially have multiple solutions, leading to the possibility of multiple equilibria. In the following of this section, we will discuss how the homotopy method can be used to compute multiple equilibria for our model of static interactions.

4.1 The homotopy method

The homotopy theory provides a set of high level conditions for the representation of a system of equations such that its solution set consists only of continuously differentiable paths and can be characterized by the system of basic differential equations. Various implementation algorithms can be developed to trace out the homotopy paths. Among the different algorithms, the homotopy continuation method (which will simply be referred to as the homotopy method in the rest of the paper) is a well known generic algorithm for looking for a fixed point to a system of nonlinear equations. A well-designed homotopy implementation algorithm is capable of finding multiple solutions of the nonlinear system, and in some cases, all solutions to the system.[3]

Our goal is to find, for all possible parameter values and realized state variables s, the solutions for the fixed point system (4.7): $\sigma - \Gamma(\sigma) = 0$. To simplify the notation, we suppress the fact that the choice probabilities depend on the state $\sigma = \sigma(s)$.

The basic idea behind the homotopy method is to take a system for which we know the solution and "transform" this system into the system that we are interested in. Formally, a homotopy is a linear mapping between the two topological spaces of functions of the form

$$H(\sigma, \tau) = \tau G(\sigma) + (1 - \tau)(\sigma - \Gamma(\sigma)), \quad \tau \in [0, 1], \tag{4.9}$$

where each of $H(\sigma, \tau)$ and $G(\sigma)$ are vectors of functions with $n \times K$ component functions: $H_{i,a_i}(\sigma, \tau)$ and $G_{i,a_i}(\sigma)$ for $i = 1, \ldots, n$ and $a_i = 1, \ldots, K$. $H(\sigma, \tau)$ is the homotopy function and τ is the homotopy parameter. Ultimately, the objective is to solve for $H(\sigma, 0)$. Varying τ from 1 to 0 "maps" the function $G(\cdot)$ into the function $\Gamma(\cdot)$. We start with $\tau = 1$ and choose $G(\sigma)$ to be a system for which it is very easy to obtain the solutions to $G(\sigma) = 0$. If for each $0 \leq \tau < 1$, we can solve for the nonlinear equations, $H(\sigma, \tau) = 0$, then by moving along the path in the direction of $\tau = 1$ to $\tau = 0$, at the end of the path we should be able to reach a solution of the original nonlinear equations $\sigma - \Gamma(\sigma) = 0$. This path then constructs a mapping between a solution of the initial system $G(\sigma) = 0$ and a solution to the fixed point problem of interest, $\sigma - \Gamma(\sigma) = 0$.

In practice, algorithms for solving differential equations can be used to trace the path from $\tau = 1$ to $\tau = 0$. In general, the solution to $H(\sigma, \tau) = 0$ forms a correspondence with finite yet multiple number of solutions at each given parameter τ. The use of complex numbers and the assumptions for theorem 1 in section 4.2 ensure that the sequence of correspondences along τ forms a set of paths along each of which the solution $\sigma(\tau)$ is uniquely defined. This is the case that we will focus in the

[3] The materials in chapters 1, 2 and 18 of Zangwill and Garcia (1981) are most closely related to the all-solution homotopty method discussed here. Information about the homotopy method can also be found in other sources including Watson, Billups, and Morgan (1987), Kostreva and Kinard (1991), Allgower and Georg (1980) and Watson, Sosonkina, Melville, Morgan, and Walker (1997).

following. From a starting point of an initial solution of a path at $\tau = 1$, we denote the solution along this particular path by $\sigma(\tau)$: $H(\sigma(\tau), \tau) = 0$. By differentiating this homotopy function with respect to τ:

$$\frac{d}{d\tau} H(\sigma(\tau), \tau) = \frac{\partial H}{\partial \tau} + \frac{\partial H}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial \tau} = 0.$$

This defines a system of differential equations for $\sigma(\tau)$ with initial condition $\sigma(1)$ calculated from the solution of the (easy) initial system $G(\sigma(1)) = 0$. In order to obtain a solution $\sigma(0)$ of the original system $\sigma - \Gamma(\sigma) = 0$, numerical algorithms of nonlinear systems of differential equations can be used to trace the path of $\tau = 1$ to $\tau = 0$. A regularity condition is necessary to ensure the path monotonicity of the system of homotopy equations. An implicit assumption in the conditions stated below is the convention of the homotopy literature that the system has to have a smooth parameterization.

Condition 1 (Regularity)**.** *Let $\nabla(\tau)$ denote the Jacobian of the homotopy functions with respect to σ at the solution path $\sigma(\tau)$:*

$$\nabla(\tau) = \frac{\partial}{\partial \sigma} Re\{H(\sigma, \tau)\}\Big|_{\sigma=\sigma(\tau)},$$

where $Re\{H(\sigma, \tau)\}$ denotes the real component of the homotopy functions. The jacobian $\nabla(\tau)$ has full rank for almost all τ.

This condition ensures the smoothness and differentiability of the paths. It rules out cases of pitchfork, transcritical bifurcation, branching and infinite spiraling. The mapping between $G(\sigma)$ and $\sigma - \Gamma(\sigma)$ is called a conformal one if the path that links them is free of these complications. If a homotopy system satisfies the regularity condition, it will either reach a solution or drift off to infinity. Intuitively, this condition prevents the the paths fromm bending back on themselves. Note that it follows from the combination of the Path Theorem and the continuous differentiability and regularity structure of the model that the solution set consists only of paths.

The all solution homotopy is one where the initial system $G(\sigma)$ is chosen such that, if we follow the paths originating from each of the solutions to $G(\sigma) = 0$, we will reach all solutions of the original system $\sigma = \Gamma(\sigma)$ at the end of the path. An all solution homotopy has to satisfy an additional path finiteness condition:

Condition 2 (Path Finiteness)**.** *Define $H^{-1}(\tau)$ to be the set of solutions $\sigma(\tau)$ to the homotopy system at τ. $H^{-1}(\tau)$ is bounded for all $0 \leq \tau < 1$. In other words, for all $\tau > 0$.*

$$\lim_{||\sigma|| \to \infty} H(\sigma, \tau) \neq 0.$$

This condition requires that the paths do not diverge for $\tau(0, 1]$. The reason that the requirement of nondivergence needs to hold at $\tau = 1$ but not at $\tau = 0$ is very intuitive. On the one hand, if the path diverges at $\tau = 0$, then some paths emanating from an initial solution will not converge to a solution of the target system. This is not a problem per se as long as paths from the solution set of the initial system lead to both all solutions of the target system and to divergence. In this case we still find all the solutions of the target system even though some of the paths diverge. On the other hand, divergence of the path at $\tau = 1$ is problematic. This implies that a solution of the target system can not be reached by a path emanating from a solution of the initial system. Therefore we might not be able to find all the solutions of the final target system. In summary, these two conditions in combination ensures that there is a unique path intersecting each solution of the target system, and likewise for the initial system. In addition, each path that intersects a solution of the target system can be traced back to a solution of the initial system.

4.2 Multiple equilibria in static discrete games

As we noted in the previous section, the issue of multiple equilibria in static interaction models amounts to the issue of computing all the fixed points to the system of equations of choice probabilities defined in equation (4.7). Note that the argument to the mapping from expected utility to choice probabilities, $\Gamma(\cdot)$, is linear in the choice probabilities of competing agents $\sigma_{-i}(a_{-i}|s)$. Therefore, the question of possible multiplicity of equilibria depends crucially on the functional form of Γ, which in turn depends exclusively on the assumed joint distribution of the error terms.

Except in a very special case of linear probability model with no corner solutions, the issue of multiple equilibria typically arises. In some models, for example if we have nonlinear interactions of the individual choice probabilities in the linear probability model, or if the joint distribution of the error term in the multinomial choice model is specified such that Γ_i is a polynomial function for each $i = 1, \ldots, n$, then all the equilibria can be found by choosing a homotopy system where the initial system of equations $G_{i,a_i}(\sigma)$ for $i = 1, \ldots, n$ and $a_i = 1, \ldots, K$. takes the following simple polynomial form:

$$G_{i,a_i}(\sigma) = \sigma_i(a_i)^{q_{i,a_i}} - 1 = 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad a_i = 1, \ldots, K, \tag{4.10}$$

where q_{i,a_i} is an integer that exceeds the degree of the polynomial of Γ_{i,a_i} as a function of $\sigma_{-i}(a_{-i})$. This results in a homotopy mapping

$$H_{i,a_i}(\sigma, \tau) = \tau\{\sigma_i(a_i)^{q_{i,a_i}} - 1\} + (1 - \tau)(\sigma_i(a_i) - \Gamma_{i,a_i}(\sigma)), \ \tau \in [0, 1]. \tag{4.11}$$

For $\tau = 0$ the system (4.9) coincides with the original system while for $\tau = 1$ it is equal to the 'simple' system (4.10).

It is a well known result from the foundation of complex analysis (see e.g. Zangwill and Garcia (1981)) that there are exactly q_{i,a_i} complex roots to $G_{i,a_i}(\sigma)$ that are evenly distributed on the unit circle. Nondegenerate polynomial functions are analytic and the regularity condition of the resulting homotopy system is automatically satisfied. The particular choice of q_{i,a_i} also ensures the path finiteness property of the homotopy system for $0 < \tau \leq 1$ (c.f. pp355-356 of Zangwill and Garcia (1981)).

While a polynomial model for $\Gamma(\cdot)$ is convenient for calculating multiple equilibria, it is rarely used in applied problems because it is not clear what parametric utility specification will give rise to a polynomial choice probability function. The most popular multinomial choice probability functions are probably the multinomial logit, the ordered logit, and the multinomial probit models. Our analysis in the following will consist of three steps. First we will establish that the incopmlete information game model in 4.7 with possible alternative error specifications has have a finite number of equilibria represented by real solutions. Second, we will show that by letting the degree of the initial polynomial system increase to infinity at an appropriate rate, the homopoty method will be able to find all the equilibria for the multinomial logit choice model (4.7). We prove it by first verifying that the homotopy mapping is regular in the complex space when the discontinuity points of the original function are isolated, and then providing a method to make homotopy work in the small vicinity of discontinuity points.

To show that the fixed point system (4.7) has a finite number of solutions in the real line, note that in general, this function is clearly continuous and infinitely differentiable with nonsingular derivatives. Therefore Brouwer's fixed point theorem will apply. It is easy to verify this condition for the multinomial logit and probit models that are commonly used in practice. Consider a compact ball B_R in $\mathbb{R}^{nK}$ with radius larger than 1. By Sard's theorem the set of irregular values of $\Gamma(\sigma) - \sigma$ has measure zero. It can be verified by differentiation through the implicit function theorem that zero is its regular value. This implies that the submanifold of σ satisfying $\Gamma(\sigma) - \sigma$ is a compact subset of this ball B_R, which contains a finite number of points. This verifies that the set of solutions in B_R in finite. Obviously, all the solutions must satisfy $0 \leq \sigma_i(a_i) \leq 1$. Therefore there can not be solutions outside B_R. However, the validity of finding all solutions by continuously increasing the order of the homopoty is mostly a theoretical results, since we are not able to provide an algorithmic stop rule to automate the determination of the order of the homotopy polynomial. In practice, we suggest a course of action to increase $q_{i,a}$ until the method can not find any more equilibria.

While we have just shown that there are in general a finite number of multiple equilibria, to compute these equilibria we need to make use of an all solution homotopy system defined in (4.11). In the following we will show that with a sufficiently high orders of the initial system q_{i,a_i}'s, a homotopy system of the form of (4.11) will find all the solution to the original system of choice probabilities. Verifying the validity of the all solution homotopy requires specifying the particular functional form of the joint distribution of the error terms in the latent utilities and checking the regularity condition and the path finiteness condition, which in turn require extension of the real homotopy system into the complex space.

The following Theorem 1 and Theorem 2 formally state this result. In the statement of the theorems, $\sigma = \{\sigma_r, \sigma_i\}$ denotes more generally a vector of the real part and the imaginary part of complex numbers which extend the real choice probabilities we considered early into the complex space. Theorem 1 first establishes the regularity properties of the homotopy outside the imaginary subspace.

THEOREM 1. *Define the sets* $H^{-1} = \{(\sigma_r, \sigma_i, \tau) \mid H(\sigma_r, \sigma_i, \tau) = 0\}$ *and*

$$H^{-1}(\tau) = \{(\sigma_r, \sigma_i) \mid H(\sigma, \tau) = 0\} \quad \textit{for} \quad \sigma_r \in \mathbb{R}^{nK}, \quad \textit{and} \quad \sigma_i \in \mathbb{R}^{nK}.$$

Note that H *is a homotopy of dimension* R^{2nK} *that include both real and imaginary parts separately. Also define, for any* ϵ *that is sufficiently small,* $\wp_\epsilon = \cup_{i,a_i}\{|\sigma_{r,i,a_i}| \leq \epsilon\}$ *to be the area around the imaginary axis, where* σ_{r,i,a_i} *denotes the component of the* σ_r *vector that corresponds to the ith player's action* a_i*. Then:*

1) *The set* $H^{-1} \cap \{\mathbb{R}^{2nK} \setminus \wp_\epsilon \times [0, 1]\}$ *consists of closed disjoint paths that do not intersect each other.*

2) *For any* $\tau \in (0, 1]$ *there exists a bounded set such that* $H^{-1}(\tau) \cap \mathbb{R}^{2nK} \setminus \wp_\epsilon$ *is in that set.*

3) *For* $(\sigma_r, \sigma_i, \tau) \in H^{-1} \cap \{\mathbb{R}^{2nK} \setminus \wp_\epsilon \times [0, 1]\}$ *the homotopy system allows parametrization*

$$H(\sigma_r(\omega), \sigma_i(\omega), \tau(\omega)) = 0.$$

Moreover, $\tau(\omega)$ *is a monotone function.*

Remark: Theorem 1 establishes the regularity and path finiteness conditions for the homotopy (4.11) in areas that are not close to the pure imaginary subspace in the complex domain $\mathbb{C}^{nK}$. The homotopy system can become irregular along the pure imaginary subspace, because the denominator in the system can approach zero and the system will become nonanalytic in the case. However, the next theorem implies that if we continue to increase the power q_{i,a_i} of the initial system (4.10) of the homotopy, we will eventually be able to find all the solutions to the original system. This also implies, however, we might lose solutions when we continue to increase q_{i,a_i}. But Theorem 2 does imply that for sufficiently large q_{i,a_i}, no new solutions will be added for larger powers. In the Monte Carlo simulation that we will report in the next section, we do find this to be the case.

THEOREM 2. *For given* τ *one can pick the power* q_{i,a_i} *of the initial function (4.10) such that the homotopy system is regular and path finite given some sequence of converging polyhedra* $\wp_\epsilon$*,* $\epsilon \to 0$*.*

5 Numerical Application

We perform several numerical simulations for an entry game with a small number of potential entrants. Player's payoff functions for each player i were constructed as linear functions of the indicator of the rival's entry $(a_i = 1)$, market covariates and a random term:

$$u_i(a_i = 1, a_{-i}) = \theta_1 - \theta_2 \left(\sum_{j \neq i} \mathbf{1}(a_j = 1) \right) + \theta_3 x_{1i} + \theta_4 x_2 + \epsilon_i(a), \ i = 1, \ldots, n.$$

The payoff of staying out is equal to $U_i(a_i = 0, a_{-i}) = \epsilon_i(a)$, where the $\epsilon_i\,(a)$ have i.i.d extreme value distributions across both a and i. The coefficients in the model are interpreted as: θ_1 is the fixed benefit of entry, θ_2 is the loss of utility when one other player enters, θ_3, θ_4 are the sensitivities of the benefit of entry to market covariates, which serve as the observed state varibles of the model.

The game can be solved to obtain *ex-ante* probabilities of entry in the market. The solution to this problem is given by:

$$P_i = \frac{e^{\theta_1 - \theta_2\left(\sum_{j \neq i} P_j\right) + \theta_3 x_{1i} + \theta_4 x_2}}{1 + e^{\theta_1 - \theta_2\left(\sum_{j \neq i} P_j\right) + \theta_3 x_{1i} + \theta_4 x_2}},\ i = 1, \ldots, n.$$

Here P_i is the *ex-ante* probability of entry for the player i, $P_i = p\,(a_i = 1|x)$. Both coefficients of the model and market covariates were taken from independent Monte-Carlo draws. The parameters of generated random variables are presented in Table 1. The means and variances of parameter values and market covariates were chosen so to have a fair percentage of cases with more then one equilibrium. For the games with 3,4 and 5 players 400 independent parameter combinations for every player were taken. The modification of the HOMPACK algorithm was run to solve for all equilibria in each game.

Before we present the simulation results, we should note that the paths that follow from Theorems 1 and 2 are drawn through the complex space and do not necessarily lead to real solutions. The existence of real solutions, however, is guaranteed by Brouwer's fixed point theorem. In general they can certainly lead to non-zero imaginary component. Therefore at least one of the complex solutions has to have a zero imaginary component. In this example the number of equilibria at each parameterization can not be determined analytically. An alternative approach is to examine a model in which the number of equilibria can be analytically determined, e.g. Grieco (2010), even though models with these analytic solutions may not be reflective of the types of games one typically solves in empirical applications. We did not pursue this alternative approach.

Throughout the Monte-Carlo runs both coefficients and covariates x_1 and x_2 were changing. So, every equilibrium was calculated for a specific set of parameters. Summary statistics for the results of computations are presented in Table 2 to 4. It is possible to see from these tables that in the constructed games the players have approximately same average entry probabilities in every type of game. While the first two entry probabilities in Table 4 are visibly lower, they are still within standard errors. This agrees with the symmetric form of underlying data generating process that players are ex ante symmetric for each simulated draw of the coefficients and market covariates.

Tables 5-7 tabulate the frequencies of different number of equilibria that are being observed in the simulations, classified according to the number of players in the market. Interestingly, a dominant number of simulations have only a single equilibrium. In addition, the frequency of observing multiple equilibria seems to decrease with the number of players in the market. In other words, we observe a larger number of multiple equilibria in the three player case but only observe a handful of them in the five player case. Complex solutions are discarded when results are reported the tables discussed below.

Tables 8-10 tabulate the probability of entry of the first player classified by the number of equilibria and the number of players in the market. In general, what we see from these tables is that there is no clear correlation pattern between the entry probability and the numbers of equilibria and players in the market.

6 Estimation with Pre-Computed Equilibria

Our technique for computing equilibria can be in principle directly used for construction of likelihood functions in cases where there are multiple equilibria compatible with the data. However, given that the provided algorithm is guaranteed to deliver all equilibria only asymptotically, it may be computationally intensive and can slow down the likelihood maximization routine. Consider a simple parametric example of entry game with an equlibrium selection mechanism based on the utilities of players. Specifically, as in our example above, we use

$$\pi_i\left(a_i, a_{-i}, s; \theta\right) = s_i'\,\beta + \delta \sum_{-i} \mathbf{1}\{a_j = 1\}$$

Suppose that $\sigma_i^*(s)$ denote equilibrium profiles and let $\mathcal{S}$ be the set of equilibria. Then the probabilities of different equilibria are determined by the choice-specific payoffs from entry

$$p(\sigma^*(s)) = \frac{\exp\left(\sum_i \nu_i\, \pi_i^*(1, s; \theta)\right)}{\sum\limits_{\sigma^{*\prime} \in \mathcal{S}} \exp\left(\sum_i \nu_i\, \pi_i^{*\prime}(1, s; \theta)\right)}.$$

In other words, the probabilities of different equilibria in the equilibrium set are determined by the weighted social welfare in the game. Then the system of probabilities of entry can be constructed as a system of conditional moments depending on the observable state variables and containing unknown parameters. As we noticed in Section 3, under the exclusion restriction, the entry probabilities obtained from the first stage can be considered as additional variables. Thus we obtain the second-stage system of conditional probabilities:

$$\Pr\left(a_i = 1 \,|\, s\right) = \sum_{\sigma^*(s) \in \mathcal{S}} p(\sigma^*(s)) \frac{\exp\left(s_i'\beta + \delta \sum_{-i} \sigma_j^*(s_j)\right)}{1 + \exp\left(s_i'\beta + \delta \sum_{-i} \sigma_j^*(s_j)\right)}, \quad \text{for} \;\; i = 1, \ldots, n.$$

Given that this equation is valid for any s, the parameters can be estimated using the vector of instrumental variables z constructed from s. Note that in this equation the conditional probability of action is directly observable from the data, while functions $\sigma_j^*(\cdot)$ need to be obtained from the computation of equilibrium solution. Then using the instrument vector z we can construct a system of unconditional moments

$$E\left[z\left(\Pr\left(a_i = 1 \,|\, s\right) - \sum_{\sigma^*(s) \in \mathcal{S}} p(\sigma^*(s)) \frac{\exp\left(s_i'\beta + \delta \sum_{-i} \sigma_j^*(s_j)\right)}{1 + \exp\left(s_i'\beta + \delta \sum_{-i} \sigma_j^*(s_j)\right)}\right)\right] = 0, \quad \text{for} \;\; i = 1, \ldots, n.$$

This is an overidentified system of moments which can be used to estimate the parameters of interest via the Generalized Method of Moments. Because of the need of solving for the equilibria for each parameter value, this method is computationally intensive.

One possible way of reducing this computational burden is to use a set of pre-computed equilibia for the same configuration of the game as in the data with "realistic" parameter values. More specifically, we can suggest the following algorithm for computing equilibria approximations. First, we simulate a sample of "realistic" parameter values and state variables. Second, we compute all equilibria in the considered draws of state variables and parameters. Third, we recover the functions corresponding to equilibrium choice probabilities as functions of state variables and parameters. In fact, the composite object

$$f(\theta, \nu, s) = \sum_{\sigma^*(s) \in \mathcal{S}} p(\sigma^*(s)) \frac{\exp\left(s_i'\beta + \delta \sum_{-i} \sigma_j^*(s_j)\right)}{1 + \exp\left(s_i'\beta + \delta \sum_{-i} \sigma_j^*(s_j)\right)}$$

is a single-valued function of state variables and parameters and can be approximated using conventional methods to obtain $\hat{f}(\cdot)$. Fourth, substituting the expectation by summation over observed sample we construct a system of empirical moments

$$\frac{1}{T} \sum_{t=1}^{T} z_t \left(\mathbf{1}\{a_{it} = 1\} - \hat{f}(s_t, \theta, \nu) \right) = 0, \quad \text{for } i = 1, \dots, n, \tag{6.12}$$

which is then used to estimate parameters θ and ν.

Alternatively, if one is willing to use a random coefficient specification for the parameters in the utility function and the equilibrium selection mechanism, then one can use the importance sampling idea in Ackerberg (2009) to construct a simulated likelihood (or a system of moments) using the distribution of state variables used for constructing the sample of equilibria. Using this method in combination with an importance sampling density, the sets of equilibria can be precomputed for the state variables observed in the sample and for the simulated draws of the utility and equilibrium selection parameters generated from the importance sampler. Then optimizing the likelihood function or the method of moment objective function with respect to the hyperparameters of the random coefficient model will only require recalculating the ratio of the density of the utility and equilibrium selection parameters when the hyperparameters change to the importance sampling density. The sets of equilibiria do not need to be recomputed for different evaluations of the hyperparameters.

7 Conclusion

In this paper we provided analysis of the all-solution homotopy method applied to compute all equilibria in the static games of incomplete information. Computation of equilibria in such games is usually a challenging task because the best-response correspondences can be highly nonlinear and the sets of equilibria can have complex configurations. The all-solution homotopy approach allows one to use a set of existing path-tracking tools offered in HOMPACK package. We provide some results that these tools can provide asymptotically good approximations to sets of equilibria in games. We illustrate our findings using a simple example of an entry game with linear payoffs. We also discuss a possible approach to estimating games with multiple equilibria using a pre-computed set of equilibria.

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A Proof for Theorem 1

In order to clarify manipulations and mathematical notations, in the following we will focus on the multinomial logit case which is the most widely used discrete choice model in the empirical literature. Similar results can be obtained for other multinomial choice models, including the ordered logit model.

Before we set out to prove the theorem we need to introduce some notations. Collapse the indexation for $i = 1, \ldots, n$ and $a_i = 1, \ldots, K$ to a single index $j = 1, \ldots, nK$. In other words, each j represents a (i, a_i) pair. First we will rewrite the expression (4.8) for the case of multinomial choice probability as:

$$\sigma_j = \frac{\exp\left(\mathcal{P}_j(\sigma)\right)}{1 + \sum_{k \in I_i} \exp\left(\mathcal{P}_k(\sigma)\right)}, \tag{A.13}$$

where $I_i = \{(i, a_i), a_i = 1, \ldots, K\}$ is the set of all indices $j = (i, a_i)$ that corresponds to the set of strategies available to player i, and $P_j(\sigma)$ is the expected utility associated with player i for playing a_i when $j = (i, a_i)$:

$$\mathcal{P}_j(\sigma) = P_{i,a_i}(\sigma) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s)\, \Phi_i(a_i, a_{-i}, s)'\,\theta,$$

which is in general a polynomial function in σ_j. Let $\mathcal{P}(\cdot)$ denote the vector-function of polynomials of size $nK \times 1$ that collects all the elements $\mathcal{P}_j(\cdot)$ for $j = 1, \ldots, nK$. Let Q be the product of the degrees of the polynomial over all elements of the vector $\mathcal{P}(\cdot)$. In other words, $Q = \prod_{j=1}^{nK} Q_j$ where Q_j is the degree of polynomial $\mathcal{P}_j(\cdot)$. For each complex argument $\xi \in \mathbb{C}^{nK}$ the system of polynomials has exactly Q solutions. Because of this, for each $\xi \in \mathbb{C}^{nK}$ we can find Q vectors σ^* such that $\mathcal{P}(\sigma^*) = \xi$. Let us denote each particular vector σ^* by $\mathcal{P}_{(k)}^{-1}(\xi)$.

The complex-valued vector $\mathcal{P}(\cdot)$ of dimension $nK \times 1$ can be transformed into a real-valued vector of dimension $2nK \times 1$ by considering separately real and complex part of vector $\mathcal{P}(\cdot)$. Because of the polynomial property, each $\mathcal{P}_{(k)}^{-1}(\xi)$ is a continuously differentiable function of ξ for almost all ξ. It is possible that for some range of the argument ξ, two (or more) solution paths $\mathcal{P}_{(k)}^{-1}(\xi)$ and $\mathcal{P}_{(k')}^{-1}(\xi)$ for $k \neq k'$ might coincide with each other. In this case we will relabel the paths k so that the merged paths create a total Q of smooth solution paths $\mathcal{P}_{(k)}^{-1}(\xi)$.

The following analysis will apply to each individual branch $\mathcal{P}_{(k)}^{-1}(\xi)$, which we will just denote by $\mathcal{P}^{-1}(\xi)$ without explicit reference to the path indice k. For $j = 1, \ldots, nK$ introduce the following notations: $\xi_j = x_j + i y_j$, $\rho_j = \|\xi_j\|$, $\varphi_j = \arctan\left(\frac{y_j}{x_j}\right)$. Then a homotopy system can be constructed for (A.13) as:

$$\begin{aligned} H_{1j}(\xi, \tau) = {} & \{\rho_j^q \cos(q\varphi_j) - 1\}\tau + (1-\tau)\left\{Re\{\mathcal{P}^{-1}(\xi)\} - \right. \\ & \left. - \frac{e^{2x_j} + e^{x_j}\cos(y_j) + \sum_{k \neq j} e^{x_j + x_k}\cos(y_j - y_k)}{1 + \sum_{k \in I_i} e^{2x_k} + 2\sum_{k \in I_i} e^{x_k}\cos(y_k) + \sum_{l \in I_i}\sum_{k \neq l} e^{x_k + x_l}\cos(y_l - y_k)}\right\} \end{aligned}$$

and

$$H_{2j}(\xi,\tau)=\{\rho_j^q\sin(q\varphi_j)-1\}\tau+(1-\tau)\left\{Im\{\mathcal{P}^{-1}(\xi)\}- -\frac{e^{x_j}\sin(y_j)+\sum_{k\neq j}e^{x_k+x_j}\sin(y_j-y_k)}{1+\sum_{k\in I_i}e^{2x_k}+2\sum_{k\in I_i}e^{x_k}\cos(y_k)+\sum_{l\in I_i}\sum_{k\neq l}e^{x_k+x_l}\cos(y_l-y_k)}\right\}$$

If the system $\mathcal{P}(\cdot)$ is polynomial, $\mathcal{P}^{-1}(\xi)$ is smooth and has a Jacobian of full rank for almost all ξ. Therefore, we can locally linearize it so that $\mathcal{P}^{-1}(\xi)\approx\Lambda\xi+\mathcal{C}$. (This expansion is used only for the purpose of clarity. A sufficient fact for the validity of the proof is that there exist Λ and $\mathcal{C}$ such that $|\mathcal{P}^{-1}(\xi)|\leq\Lambda|\xi|+\mathcal{C}$ which is true if $\mathcal{P}(\cdot)$ is a polynomial.) The homotopy system can then be written as:

$$H_{1j}(\xi,\tau)=\{\rho_j^q\cos(q\varphi_j)-1\}\tau+(1-\tau)\left\{\Lambda^j x_j- -\frac{e^{2x_j}+e^{x_j}\cos(y_j)+\sum_{k\neq j}e^{x_j+x_k}\cos(y_j-y_k)}{1+\sum_{k\in I_i}e^{2x_k}+2\sum_{k\in I_i}e^{x_k}\cos(y_k)+\sum_{l\in I_i}\sum_{k\neq l}e^{x_k+x_l}\cos(y_l-y_k)}\right\} \tag{A.14}$$

and

$$H_{2j}(\xi,\tau)=\{\rho_j^q\sin(q\varphi_j)-1\}\tau+(1-\tau)\left\{\Lambda^j y_j- -\frac{e^{x_j}\sin(y_j)+\sum_{k\neq j}e^{x_k+x_j}\sin(y_j-y_k)}{1+\sum_{k\in I_i}e^{2x_k}+2\sum_{k\in I_i}e^{x_k}\cos(y_k)+\sum_{l\in I_i}\sum_{k\neq l}e^{x_k+x_l}\cos(y_l-y_k)}\right\} \tag{A.15}$$

where Λ^j is the jth row of the $nK\times nK$ matrix Λ. Without loss of generality we will let $\mathcal{C}=0$ in subsequent analysis for the sake of brevity because all the results will hold for any other given $\mathcal{C}$. To simplify notation we will denote:

$$\Theta_i(x,y)=\sum_{k\in I_i}e^{2x_k}+2\sum_{k\in I_i}e^{x_k}\cos(y_k)+\sum_{l\in I_i}\sum_{k\neq l}e^{x_k+x_l}\cos(y_l-y_k)$$

Now given some index $k\in\{1,\ldots,Q\}$, we consider the solutions of the system $\{H(x,y,\tau)\}=0$ for all possible real values of the vectors of x and y.

Now we set out to prove the statements of Theorem 1. First we will prove statement (2). Define $\rho=\|\xi\|$ to be the Euclidean norm of the entire $nK\times 1$ vector ξ. We need to prove that there will not be a sequence of solutions along a path where $\rho\to\infty$. We will show this by contradiction. Consider a path where $\rho\to\infty$. Choose the component j of the homotopy system for which $\rho_j^q\cos(q\,\varphi_j)\to\infty$ at the fastest rate among all the possible indexes j where $\rho_j\to\infty$. [4]

Consider the real part of the homotopy function, $H_{1j}(\cdot,\cdot,\cdot)$. The equation $H_{1j}(x,y,\tau)=0$ is equivalent to the equation $\frac{H_{1j}(x,y,\tau)}{\tau\left(\rho_j^q\cos(q\,\varphi_j)-1\right)}=0$ for $\rho_j>1$. The last equation can be rewritten as:

$$1+\frac{(1-\tau)}{\tau\left(\rho_j^q\cos(q\,\varphi_j)-1\right)}\left\{\Lambda^j x-\frac{e^{2x_j}+e^{x_j}\cos(y_j)+\sum_{k\neq j}e^{x_j+x_k}\cos(y_j-y_k)}{1+\Theta_i(x,y)}\right\}=0. \tag{A.16}$$

[4] In case when instead of $\rho_j^q\cos(q\,\varphi_j)\to\infty$ we have that $\rho_j^q\sin(q\,\varphi_j)\to\infty$, the proof can be appropriately modified by considering the imaginary part of the j-th element of the homotopy system without any further changes. The logic of the proof can be seen to hold as long as there is a slower growing element of x or y. In case when all components of x and y grow at the same rate to infinity in such a way that the second terms inside the curly brackets of (A.14) and (A.15) explode to infinity, one can take a Laurent expansion around the values of y_k's such that the denominators are close to zero. Then one can see that these terms in (A.14) and (A.15) explode to infinity at quadratic and linear rates in $1/(y-y^*)$, respectively. Therefore (A.14) and (A.15) can not both be zero simultaneously for large x and y.

We will show that the second term in the curly bracket of the previous equation is uniformly bounded from above in absolute terms:

$$\left|\frac{e^{2x_j} + e^{x_j}\cos(y_j) + \sum_{k\neq j} e^{x_j+x_k}\cos(y_j - y_k)}{1+\Theta_i(x,y)}\right| \leq C \quad \text{and} \quad \text{for a constant } C, \tag{A.17}$$

where the constant C can depend on ϵ. Therefore the term in the curly bracket in the homotopy (A.16) will grow at most at a linear rate $|x| \leq C\rho_j$. On the other hand, denominator $\tau\left(\rho_j^q \cos(q\varphi_j) - 1\right)$ outside the curly bracket grows at a much faster polynomial rate for large q. Hence the second term in (A.16) is close to 0 for large q for large values of ξ, and equation (A.16) can not have a sequence of solutions that tends to infinity.

In other words, there exists $R_0 > 0$ such that for any $\xi = (x, y)$ outside $\wp_\epsilon$ with $\|\xi\| \geq R_0$ and any $\tau \in (0, 1]$ we have that $H_1(x, y, \tau) \neq 0$, that is, homotopy system does not have solutions. This implies that

$$H^{-1}(\tau) \cap \wp_\epsilon \subset B^\tau_{R_0} = \{(x, y, \tau) \in \mathbb{R}^{2nK} \setminus \wp_\epsilon \times (0, 1] \cap \|\xi\| < R_0\}.$$

This proves the statement 2).

Finally, we will prove both statements 1) and 3) of Theorem 1. Again we consider the above homotopy system on the compact set $B^\tau_{R_0}$. The homotopy function is analytic in this set so Cauchy - Riehmann theorem holds. This implies that

$$\frac{\partial H_{1j}}{\partial x_k} = \frac{\partial H_{2j}}{\partial y_k} \quad \text{and} \quad \frac{\partial H_{1j}}{\partial y_k} = -\frac{\partial H_{2j}}{\partial x_k}, \qquad \text{for all } j, k = 1, \ldots, 2nK.$$

This means that if the Jacobian is considered:

$$J_i = \begin{pmatrix} \frac{\partial H_{1j}}{\partial x'} & \frac{\partial H_{1j}}{\partial y'} & \frac{\partial H_{1j}}{\partial \tau} \\ \frac{\partial H_{2j}}{\partial x'} & \frac{\partial H_{2j}}{\partial y'} & \frac{\partial H_{2j}}{\partial \tau} \end{pmatrix},$$

then it contains at least one 2×2 submatrix with nonnegative determinant $\left[\frac{\partial H_{1j}}{\partial x_k}\right]^2 + \left[\frac{\partial H_{1j}}{\partial y_k}\right]^2$. Calculating the derivatives directly due to the fact that $\epsilon < \rho < R_0$ this determinant is strictly positive for all $(x, y, \tau) \in B^\tau_{R_0}$. Therefore, the implicit function theorem verifies that the pair (x, y) can be locally parameterized by τ. Moreover, this representation is locally unique and continuous. This proves the first statement. The same arguments above, which show that the determinant is positively almost everywhere, also immediately implies the third statement. □

Proof of equation (A.17):

We are to bound the left hand side of equation (A.17) by a given constant. First of all we can bound the denominator from below by

$$\|1 + \Theta_i(x, y)\| \geq \left\| 1 + \sum_{k\in I_i} e^{2x_k} \right\| - \left\| 2\sum_{k\in I_i} e^{x_k}\cos(y_k) + \sum_{l\in I_i}\sum_{k\neq l} e^{x_k+x_l}\cos(y_l - y_k) \right\|,$$

as $\|a + b\| \geq \|a\| - \|b\|$. Then we can continue to bound:

$$\|1 + \Theta_i(x, y)\| \geq 1 + \textstyle\sum_{k\in I_i} e^{2x_k} - 2\sum_{k\in I_i} e^{x_k} - \sum_{l\in I_i}\sum_{k\neq l} e^{x_k+x_l}. \tag{A.18}$$

The last expression was obtained taking into account the fact that

$$\max_{y_k,\ k\in I_i} \|2\sum_{k\in I_i} e^{x_k}\cos(y_k) + \sum_{l\in I_i}\sum_{k\neq l} e^{x_k+x_l}\cos(y_l - y_k)\|$$

is attained at the point $\cos(y_k) \equiv \cos(y_k - y_l) = 1, \forall k, l \in I_i$.

For the same reason, we can bound the numerator from above by

$$\begin{aligned}&\left\|e^{2x_j} + e^{x_j}\cos(y_j) + \textstyle\sum_{k\neq j} e^{x_j+x_k}\cos(y_j - y_k)\right\| \leq e^{2x_j} + \|e^{x_j}\cos(y_j)\\ &+ \textstyle\sum_{k\neq j} e^{x_j+x_k}\cos(y_j - y_k)\right\| \leq e^{2x_j} + e^{x_j} + \textstyle\sum_{k\neq j} e^{x_j+x_k}.\end{aligned}$$

Recall that j - th component was assumed to be the fastest growing x component as $\rho \to \infty$. Then from equation (A.18) for some small but positive constant ψ we can write:

$$\|1 + \Theta_i(x, y)\| \geq 1 + \psi e^{2x_j}$$

Collecting terms we have that:

$$\frac{\left\|e^{2x_j} + e^{x_j}\cos(y_j) + \sum_{k\neq j} e^{x_j+x_k}\cos(y_j - y_k)\right\|}{\|1 + \Theta_i(x, y)\|} \leq \frac{1 + e^{2x_j} + e^{x_j} + \sum_{k\neq j} e^{x_j+x_k}}{1 + \psi e^{2x_j}}.$$

which is clearly uniformly bounded from above by a large constant.

The same arguments can be used by looking at the imaginary part of the homotopy system when there exists a y_j that converges to infinity at the fastest rate. □

B Proof for Theorem 2

For the clarify of exposition we will present the proof in the case of two strategies for each player. In the case with more than two strategies for each player, the expansions for the homotopy system will be more complex and will involve more terms in the denominator. But the proof strategy is very similar, except it involves more points around which expansions have to be taken.

In the two strategy case, we can rewrite the homotopy system (A.14) and (A.15) as

$$H_{1j}(\xi, \tau) = \{\rho_j^q \cos(q\varphi_j) - 1\}\tau + (1 - \tau)\left\{\Lambda^j x_j - \frac{e^{2x_j} + e^{x_j}\cos(y_j)}{1 + e^{2x_j} + 2e^{x_j}\cos(y_j)}\right\},$$

and

$$H_{2j}(\xi, \tau) = \{\rho_j^q \sin(q\varphi_j) - 1\}\tau + (1 - \tau)\left\{\Lambda^j y_j - \frac{e^{x_j}\sin(y_j)}{1 + e^{2x_i} + 2e^{x_j}\cos(y_j)}\right\}.$$

We need to check the presence of solutions in the small vicinity of the imaginary axis. Now consider positive increments of x_j such that x_j is equal to some small value ϵ. If we linear the above homotopy system around $x_j = 0$, we can approximate them linearly by

$$\begin{aligned} H_{1j} &= \tau q\epsilon y_j^{q-1} - \frac{(1-\tau)\epsilon}{2}\frac{1}{1+\cos(y_j)} - \frac{1+\tau}{2} + \lambda_{jj}(1-\tau)\epsilon + \sum_{k\neq j}\lambda_{jk}x_k(1-\tau) \\ H_{2j} &= \tau y_j^q + (1-\tau)\sum_k \lambda_{jk}y_k - \frac{1-\tau}{2}\frac{sin(y_j)}{1+cos(y_j)} - \tau \end{aligned} \tag{B.19}$$

where λ_{jj} is the j,jth element of the Λ matrix.

One can see that these two functions are continuous everywhere except for the set of points $\{y_j = \pi+2\pi k,\ k \in \mathbb{Z}\}$ where $cos\,(y_j) = -1$.

We will prove that for appropriate large values of q this system has no solutions in the vicinity of this set. First of all note that if we take a second order expansion of $1 + cos\,(y_j)$ around some $y_j^* = \pi + 2\pi k$ we can approximate $1 + cos\,(y_j) \approx \frac{1}{2}\left(y_j - y_j^*\right)^2$. Then we can further linearize these two equations in (B.19) to:

$$\begin{aligned} H_{1j} =& \tau q \epsilon {y_j^*}^{\,q-1} - \lambda_{jj}(1-\tau)\epsilon + \sum_{k\neq j} \lambda_{kj} x_k (1-\tau) - \frac{1+\tau}{2} - \frac{(1-\tau)\epsilon}{(y_j - y_j^*)^2} \\ H_{2j} =& \tau {y_j^*}^{\,q} + (1-\tau)\lambda_{jj} y_j^* - \sum_{k\neq j} \lambda_{kj} y_k (1-\tau) - \tau + (1-\tau)\frac{1}{(y_i - y_i^*)} \end{aligned} \tag{B.20}$$

where we have also used $sin\,(y_j) \approx -\left(y_j - y_j^*\right)$.

Now we can construct a sequence of homotopies with the order q increasing to infinity at appropriate rate such that these homotopies do not have solutions with extraneous solution of $|y_j| \to \infty$. This sequence of q is constructed by letting $q = 1 + 1/\epsilon$, as $\epsilon \to 0$. Along this sequence, we will see below that the solutions $y_j - y_j^*$ to H_{1j} and H_{2j} will be of different orders of magnitude. Therefore there can not solutions $y_j - y_j^*$ that simultaneously satisfy both equations $H_{1j} = 0$ and $H_{2j} = 0$.

To see this, consider the first part $H_{1j} = 0$ of (B.20). For small ϵ only the first term $\tau q \epsilon {y_j^*}^{\,q-1} = O\left({y_j^*}^{\frac{1}{\epsilon}}\right)$ and the last term $\frac{(1-\tau)\epsilon}{(y_j - y_j^*)^2}$ dominate. Therefore the solution $y_j - y_j^*$ has to have the order of magnitude $O\left(\sqrt{\frac{1}{\epsilon}}{y_j^*}^{-\frac{1}{2\epsilon}}\right)$. On the other hand, for the second part $H_{2j} = 0$ of (B.20). For small ϵ only the first term $\tau {y_j^*}^{\,q} = O\left({y_j^*}^{\frac{1}{\epsilon}}\right)$ and the last term $(1-\tau)\frac{1}{(y_i - y_i^*)}$ dominate. Therefore the solution $y_j - y_j^*$ has to have the order of magnitude $O\left({y_j^*}^{-\frac{1}{\epsilon}}\right)$ which increases to ∞ much slower than $O\left(\sqrt{\frac{1}{\epsilon}}{y_j^*}^{-\frac{1}{2\epsilon}}\right)$ as $\epsilon \to \infty$. Therefore there can be no solution y_j to both H_{1j} and H_{2j} simultaneously for the sequence of q chosen above. This proves that the homotopy is path finite along that sequence of q.

The considered homotopy function is analytic outside the balls of fixed radius around the members of countable set of points $\{x_j = 0, y_j = \pi + 2\pi k\},\ k \in \mathbb{Z}$ [5]. Therefore a monotone smooth parametrization is available except for the interior of these balls because the determinant of the Jacobian is strictly positive everywhere else.

This establishes regularity of the homotopy and concludes the proof. □

Proof of equation (B.19): We consider each term individually. First of all

$$\varphi = \arctan\,(y/\epsilon) = \frac{\pi}{2} - \arctan\,(\epsilon/y) \approx \frac{\pi}{2} - \frac{\epsilon}{y}.$$

[5] Moreover, it is possible to check that the homotopy system has no solutions when all arguments are purely imaginary in case if q is an arbitrary odd number

Hence, as long as q is chosen so that $q\pi/2$ is $2k\pi + \frac{\pi}{2}$ for some k,

$$cos\,(q\varphi) = cos\,(q \arctan{(y/\epsilon)}) \approx cos\left(q\frac{\pi}{2} - q\frac{\epsilon}{y}\right) = sin\left(\frac{q\epsilon}{y}\right) \approx \frac{q\epsilon}{y}.$$

Together with $\rho^q \approx y_j^q$, this gives the first term in H_{1j}.

Secondly, a first order expansion around $\epsilon = 0$ gives

$$\frac{e^{x_j}\sin(y_j)}{1 + e^{2x_i} + 2e^{x_j}\cos(y_j)} \approx \frac{1}{2} + \frac{1}{2}\epsilon\frac{1}{1 + cos\,(y_j)}.$$

Therefore the H_{1j} is proved in (B.19).

The second part of H_{2j} follows similarly, noting that given the choice of q where $sin(q\varphi) = 1$, and $\rho_j^q \approx y_j^q$, and the first Taylor expansion term for $\frac{e^{x_j}\sin(y_j)}{1+e^{2x_i}+2e^{x_j}\cos(y_j)}$ vanishes.

End of proof for equation (B.19).

C Tables

Table 1: Characteristics of the Parameters

PARAMETER	MEAN	VARIANCE	DISTRIBUTION
θ_1	2.45	1	Normal
θ_2	5.0	1	Normal
θ_3	1.0	1	Normal
θ_4	-1.0	1	Normal
X_1	1.0	0.33	Uniform
X_2	1.0	0.33	Uniform
$\theta_3 X_1 + \theta_4 X_2$	0.0	3.32	Mixture

Table 2: Results of Monte-Carlo Simulations, n=3

Parameter	Mean	Std. Dev	Max	Min
# of equilibria	1.592	1.175	7	1
P1	0.366	0.362	0.998	0
P2	0.36	0.367	0.995	0
P3	0.363	0.348	0.993	0.003

Table 3: Results of Monte-Carlo Simulations, n=4

Parameter	Mean	Std. Dev	Max	Min
# of equilibria	1.292	0.777	5	1
P1	0.278	0.328	0.981	0.001
P2	0.246	0.32	0.981	0.003
P3	0.276	0.338	0.999	0.001
P4	0.28	0.338	0.987	0.002

Table 4: Results of Monte-Carlo Simulations, n=5

Parameter	Mean	Std. Dev	Max	Min
# of equilibria	1.106	0.505	5	1
P1	0.104	0.201	0.964	0
P2	0.138	0.252	0.975	0
P3	0.315	0.338	0.992	0
P4	0.356	0.385	0.983	0
P5	0.319	0.344	0.982	0

Table 5: Frequencies for the number of equilibria, n=3

# of equilibria	Number of Cases	Frequency (%)
m=1	192	47.93
m=3	132	33.06
m=5	64	16.12
m=7	12	2.89
Total	400	1

D Further illustrations

In this section we provide more illustrations of the two player case for the entry game example in the the numerical application section 5. For the two player case of that example, the system of equations characterizing the Bayes-Nash equilibrium takes the form

$$F_1(\sigma_1,\sigma_2)=\sigma_1-\frac{\exp\left(\bar{\theta}_1+\bar{\theta}_2\sigma_2\right)}{1+\exp\left(\bar{\theta}_1+\bar{\theta}_2\sigma_2\right)},\quad F_2(\sigma_1,\sigma_2)=\sigma_2-\frac{\exp\left(\bar{\theta}_1+\bar{\theta}_2\sigma_1\right)}{1+\exp\left(\bar{\theta}_1+\bar{\theta}_2\sigma_1\right)},$$

because the structure of the model can be summarized by two parameters $\bar{\theta}_1=\theta_1+\theta_3x_1+\theta_4x_2$ and $\bar{\theta}_2=-\theta_2$.

To illustrate the equilibrium structure of this model, we plot the dependence of the number of equilibria

Table 6: Frequencies for the number of equilibria, n=4

# of equilibria	Number of Cases	Frequency (%)
m=1	287	71.84
m=3	93	23.3
m=5	20	4.85
Total	400	1

Table 7: Frequencies for the number of equilibria, n=5

# of equilibria	Number of Cases	Frequency (%)
m=1	373	93.16
m=3	25	6.21
m=5	2	0.62
Total	400	1

Table 8: Entry Probability of First Player, n=3

# of equilibria	Mean	STD
m=1	0.375	0.386
m=3	0.337	0.341
m=5	0.353	0.322
m=7	0.601	0.367

Table 9: Entry Probability of First Player, n=4

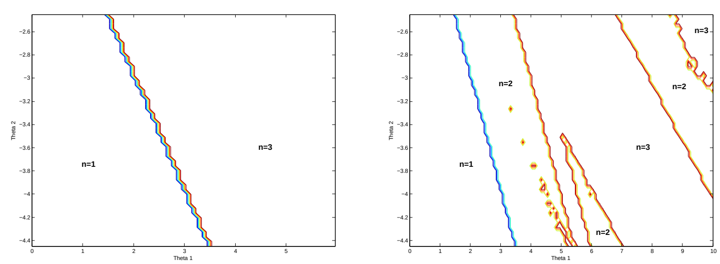
# of equilibria	Mean	STD
m=1	0.211	0.3
m=3	0.431	0.328
m=5	0.129	0.235

found using the all-solution homotopy as a function of the parameters $\bar{\theta}_1$ and $\bar{\theta}_2$ One can see a jump in the number of found equilibria from 1 to 3 with the change of the parameter values. To compare the computational results we also build a natural homotopy. For the natural homotopy we use the same system but it is constructed for the real-valued arguments. As an initial system we use a simple system $z_1 = a_1$ and $z_2 = a_2$. We select 100 different initial systems for each parameter pair $\left(\bar{\theta}_1, \bar{\theta}_2\right)$, and select the initial points

Table 10: Entry Probability of First Player, n=5

# of equilibria	Mean	STD
m=1	0.116	0.216
m=3	0.08	0.206
m=5	0.007	0.232

Figure 1: Number of solutions for all-solutions (a) and natural (b) homotopies



(a) (b)

uniformly over the square $(\sigma_1, \sigma_2) \in [0,1] \times [0,1]$. One can see that the all-solution homotopy is capable of finding all three solutions of the equilibrium system. The natural homotopy only finds 2 solutions for a large combination of parameter values. This illustrates the inferior behavior of using natural homotopy for computing the Bayes-Nash equilibria in this particular example.

We also analyzed how the behavior of the all-solution homotopy depends on the degree of the initial system. At the parameter values $\bar{\theta}_1 = 5$ and $\bar{\theta}_2 = -2.45$ where the considered equilibrium system has three solutions, We found that when $q = 2$, the all-solution homotopy finds 2 roots of the equilibrium system corresponding to the stable Bayes-Nash Equilibria. However, when the degree of the initial system increases to $q = 3$, the all-solution homotopy finds all three roots. This leads us to conclude that the behavior of the all-solution homotopy for the logistic system of two equations resembles the behavior of the all-solution homotopy for polynomials.

In the following we provide a more detailed discussion of the homotopy construction for the 2-player case. Readers not interested in these details may skip the remaining materials. To construct the homotopy system we first re-defined the original system of equations in the complex domain. To do that we make and additional change of variable $z_1 = \left(\sigma_1 - \bar{\theta}_1\right)/\bar{\theta}_2$ and $z_2 = \left(\sigma_2 - \bar{\theta}_1\right)/\bar{\theta}_2$. Then we set z_1 and z_2 to be complex such that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where $i = \sqrt{-1}$. This produces the complex-valued system of equations from the equilibrium system. We separate real and complex parts of this system creating the system with 4 unknown variables (corresponding to the real and imaginary parts of z_1 and z_2) that consists of 4 equations corresponding to the real and imaginary parts of the equilibrium system. We recall the notion of hyperbolic sine and cosine functions: $sh\, x = 0.5(e^x - e^{-x})$ and $ch\, x = 0.5(e^x + e^{-x})$ and write the real and imaginary parts of the system of equations under consideration as:

$$\begin{aligned}
\mathrm{Re}\,(F_1) &= \frac{x_1 - \theta_1}{\theta_2} - \frac{1}{2}\,\frac{\exp{(x_2)} + \cos(y_2)}{\mathrm{ch}\,(x_2) + \cos(y_2)},\\
\mathrm{Re}\,(F_2) &= \frac{x_2 - \theta_1}{\theta_2} - \frac{1}{2}\,\frac{\exp{(x_1)} + \cos(y_1)}{\mathrm{ch}\,(x_1) + \cos(y_1)}.\\
\mathrm{Im}\,(F_1) &= \frac{y_1}{\theta_2} - \frac{1}{2}\,\frac{\sin(y_2)}{\mathrm{ch}\,(x_2) + \cos(y_2)},\\
\mathrm{Im}\,(F_2) &= \frac{y_2}{\theta_2} - \frac{1}{2}\,\frac{\sin(y_1)}{\mathrm{ch}\,(x_1) + \cos(y_1)}.
\end{aligned}$$

We denote the vector-function containing the elements of this system $F(x_1, x_2, y_1, y_2)$ and we aim at finding all solutions of the system $F(x_1, x_2, y_1, y_2) = 0$. To solve this system we define an initial system of equations

$$\begin{aligned}
G_1(z_1, z_2) &= z_1^q - 1,\\
G_2(z_1, z_2) &= z_2^q - 1.
\end{aligned}$$

We note that equation $z_1^q = 1$ has q different solutions in the complex domain. In fact, given that $e^{2\pi i n} = 1$ for $n = 1, 2, \ldots$, the solutions will take the form $z_1 = e^{2\pi i n/q}$. They will be geometrically different if $1 \le n \le q$, creating q different roots. We can represent the complex-valued system by the real-valued system containing its real and imaginary parts as functions of real and imaginary arguments. As a result, we can define the initial system as

$$\begin{aligned}
\mathrm{Re}\,(G_1(z_1, z_2)) &= \left(x_1^2 + y_1^2\right)^{q/2} \cos\left(q\,\mathrm{arctg}\left(\frac{x_1}{y_1}\right)\right),\\
\mathrm{Re}\,(G_2(z_1, z_2)) &= \left(x_2^2 + y_2^2\right)^{q/2} \cos\left(q\,\mathrm{arctg}\left(\frac{x_2}{y_2}\right)\right),\\
\mathrm{Im}\,(G_1(z_1, z_2)) &= \left(x_1^2 + y_1^2\right)^{q/2} \sin\left(q\,\mathrm{arctg}\left(\frac{x_1}{y_1}\right)\right),\\
\mathrm{Im}\,(G_1(z_1, z_2)) &= \left(x_2^2 + y_2^2\right)^{q/2} \sin\left(q\,\mathrm{arctg}\left(\frac{x_2}{y_2}\right)\right).
\end{aligned}$$

We denote this system $G(x_1, x_2, y_1, y_2)$. Then the homotopy system takes the form

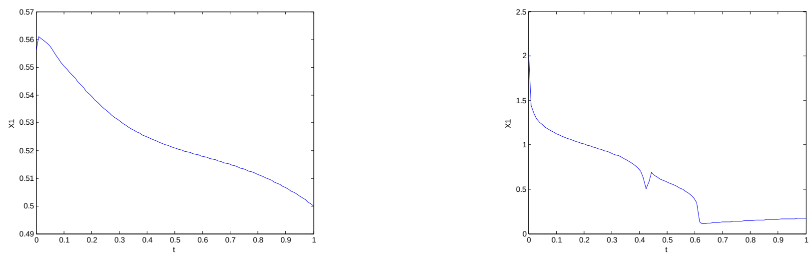
$$(1 - \tau)\, G(x_1, x_2, y_1, y_2) + \tau\, F(x_1, x_2, y_1, y_2) = 0.$$

We then construct the system of differential equations

$$\left[(1 - \tau)\,\frac{\partial G}{\partial x} + \tau \frac{\partial F}{\partial x}\ \ ;\ \ (1 - \tau)\,\frac{\partial G}{\partial y} + \tau \frac{\partial F}{\partial y}\right] \left(\begin{array}{c} \dot{x} \\ \dot{y} \end{array}\right) = F(x, y) - G(x, y).$$

This system can be solved numerically if we initialize it at $\tau = 0$ using the solutions of the "simple" system of equations with $\phi_{1,2} = \frac{2\pi n}{q}$ for $n = 0, 1, \ldots, q - 1$ such that $x_1 = \cos(\phi_1)$, $x_2 = \cos(\phi_2)$, $y_1 = \sin(\phi_1)$, and $y_2 = \sin(\phi_2)$. We then track the paths corresponding to the solutions and select the paths that lead to purely real roots. On Figure 2 we illustrate the behavior of the paths that lead to the purely real root (a)

Figure 2: Homotopy path for the first dimension as a function of τ for (a) purely real solution and (b) complex solution



(a) (b)

and a complex root (b). We display the path for the first argument of our system of interest as a function of the homotopy parameter. An advantage of the system for 2 players is that we can explicitly solve for the points of irregularity that correspond to the case where the matrix of first derivatives of the homotopy system is not invertible at $\tau = 1$. The point of irregularity correspond to the case where the denominator in the functions of the system $F(x, y) = 0$ becomes equal to zero. Note that the denominator corresponding to the first and the third equations is equal to $\cos(y_2) + \operatorname{ch}(x_2)$ and the denominator of the second and fourth equations is equal to $\cos(y_1) + \operatorname{ch}(x_1)$. When $(x_1, x_2) \in [0, 1]^2$ and $(y_1, y_2) \in [-\pi, \pi]^2$, one of the denominators is equal to zero if and only if $x_1 = y_1 = 0$ or $x_2 = y_2 = 0$. Thus the irregularity points will be avoided if the homotopy path does not cross the origin.